THE ALEXANDER POLYNOMIAL OF A TORUS KNOT WITH TWISTS

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ABSTRACT

This note gives an explicit calculation of the doubly infinite sequence $\Delta(p,q,2m), m \in \mathbf{Z}$ of Alexander polynomials of the (p,q) torus knot with m extra full twists on two adjacent strings, where p and q are both positive. The knots can be presented as the closure of the p-string braids $(\delta_p)^q\sigma_1^{2m}$, where $\delta_p = \sigma_{p-1}\sigma_{p-2}\ldots\sigma_2\sigma_1$, or equally of the q-string braids $(\delta_q)^p\sigma_1^{2m}$. As an application we give conditions on (p,q) which ensure that all the polynomials $\Delta(p,q,2m)$ with $|m| \geq 2$ have at least one coefficient a with $|a| > 1$. A theorem of Ozsvath and Szabo then ensures that no lens space can arise by Dehn surgery on any of these knots. The calculations depend on finding a formula for the multivariable Alexander polynomial of the 3-component link consisting of the torus knot with twists and the two core curves of the complementary solid tori.
Keywords: torus knot, twist, Dehn surgery, multi-variable Alexander polynomial.

1 Introduction

The calculations for the sequence $\Delta(p,q,2m), m \in \mathbf{Z}$ of Alexander polynomials of the (p,q) torus knot with m extra full twists on two adjacent strings were initially done for the $(7,17)$ torus knot in response to a query of Yoav Moriah [4] about their Alexander polynomials. The results in this case allowed him to deduce, from work of Ozsvath and Szabo [7], that the only knots in this sequence which can give a lens space after Dehn surgery are those with $m = 0, \pm 1$.

In his thesis [1] and a subsequent paper [2] John Dean studies a more general class of knots lying on the surface of a standard genus 2 surface, which he calls *twisted torus knots*. He gives a condition, which he terms *primitive/Seifert fibred*, on the knot in relation to the two complementary handlebodies. Knots satisfying this condition yield small Seifert fibre spaces (with base S^2 and at most 3 exceptional fibres) under some Dehn surgery. The knots considered in this paper are simple examples of Dean's twisted torus knots, which are primitive/Seifert fibred only in the cases $m = \pm 1$ or $q = 3$ or $q = \pm 2 \bmod p$.

My original method for the $(7, 17)$ calculation was simply to use the skein relation for the Conway polynomial to produce a recursive relation for the Conway polynomials $f_k(z)$ of any sequence of knots differing only in having k half twists at one spot in two directly oriented strands.

In the Conway skein a single half-twist σ satisfies the quadratic equation

$$\sigma^2 = z\sigma + 1$$

with roots $s, -s^{-1}$, where $s - s^{-1} = z$. This leads to the relation

$$f_{k+2} = (s - s^{-1}) f_{k+1} + f_k.$$

Solving the recurrence relation gives a formula $f_k = cs^k + d(-s)^{-k}$ in terms of s, where c and d are rational functions to be determined; the Alexander polynomial is given by setting $s^2 = t$.

Knowing the Alexander polynomials for say $k = 0$ and $k = 2$ determines c and d, and hence the whole sequence of Alexander polynomials (by setting $s^2 = t$). For the case of $(7, 17)$ an explicit Maple calculation of f_0 and f_2 was enough to find the sequence and to answer Moriah's original question.

2 Use of the reduced Burau matrix

Attempts to simplify and generalise the calculations led first to the corresponding recurrence formula for the suitably normalised multivariable Alexander polynomial a_k of a sequence of links with several components, differing by k half twists in two directly oriented strands. Where the two strands involved in the twisting belong to components both labelled with the same variable $t = s^2$ the polynomials again satisfy a recurrence relation with solution $a_k = cs^k + d(-s)^{-k}$ for some rational functions c and d determined by a_0 and a_1. This relation holds for the properly normalised form of the Alexander polynomial, as given for example by Murakami [6]. Frequently, however, the Alexander polynomial has been multiplied by a power of the variables, and a variant of this relation may work systematically.

One such variant occurs naturally when the multivariable polynomial of a closed n-braid $\hat{\beta}$ and its axis A is realised as the characteristic polynomial of the reduced Burau matrix of β, as in [5]. We can assume that the sequence of links is presented as the closure of a sequence of braids $\beta\sigma_1^k$, in which the twists take place in the first two strands, both labelled by the same meridian element t. In this representation the reduced Burau matrix for σ_1 is the $(n-1) \times (n-1)$ block matrix

$$S = \begin{pmatrix} -t & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-3},$$

which has eigenvalues $-t$ once and 1 repeated $n - 2$ times. It satisfies the equation $S^2 = (1 - t)S + tI$.

Let B be the reduced multivariable Burau matrix of β. Then BS^k is the reduced Burau matrix of $\beta\sigma_1^k$, and

$$BS^{k+2} = (1 - t)BS^{k+1} + tBS^k.$$

Since the exterior powers of S all have the two eigenvalues 1 and $-t$, and characteristic polynomials are formed by taking traces of exterior powers it follows that the polynomials $\Delta_k = \det(I - xBS^k)$ also satisfy the recurrence relation

$$\Delta_{k+2} = (1-t)\Delta_{k+1} + t\Delta_k.$$

This gives the formula

$$\Delta_{k+1} - \Delta_k = (-t)^k(\Delta_1 - \Delta_0),$$

and hence

$$\Delta_k = (1 - t + t^2 - \cdots + (-t)^{k-1})(\Delta_1 - \Delta_0).$$

For the case of $k = 2m$, with $m \geq 0$ full twists, this will also give a recurrence relation leading to the formula

$$\Delta_{2m} = (1 + t^2 + \cdots + t^{2m-2})(\Delta_2 - \Delta_0)$$

for the multivariable polynomials of the sequence of links.

3 The multivariable Alexander polynomial

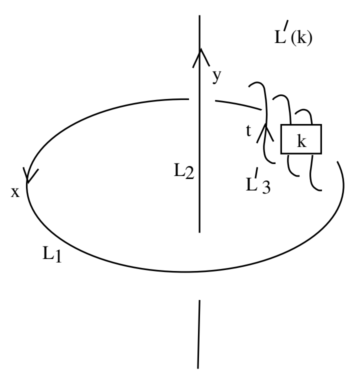
Use of the multivariable Alexander polynomial can be taken a stage further, by the application of two basic principles, due essentially to Torres [8] and Fox [3].

Suppose that L is an oriented link with several components, $L_1, \ldots, L_n$. Write $H_1(S^3 - L) \cong (C_\infty)^n$ multiplicatively, with positive meridian generator t_i corresponding to the component L_i. The Alexander polynomial Δ_L is an element of the group ring $\mathbf{Z}[H_1(S^3 - L)]$, in other words, a Laurent polynomial in $t_1, \ldots, t_n$.

Theorem 1 (Fox) *If $f : S^3 - L \to S^3 - L'$ is a homeomorphism of link exteriors, and f_* is the induced map on H_1 then*

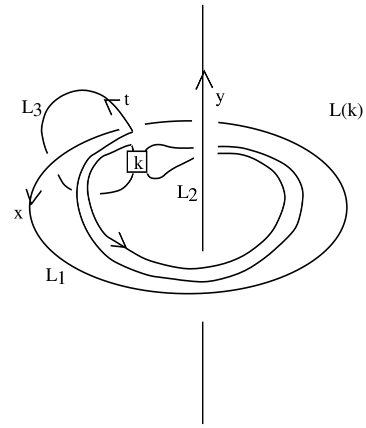
$$\Delta_{L'} = f_*(\Delta_L).$$

We want to find the Alexander polynomials of the sequence of links $L'(k)$ shown here, which consist of the (p,q) torus knot with k inserted half-twists lying on or near a standard torus T, along with the core curves L_1 and L_2 of each complementary solid torus.

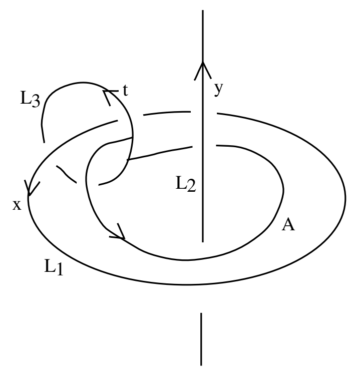


We label the meridians of the components by t, x and y as shown.

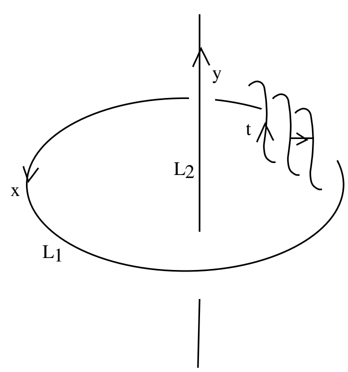
Now apply theorem 1 to the sequence of links $L(k)$, shown below,



after choosing an orientation preserving homeomorphism f of the complement of the core curves which carries T to itself and takes $L_3(k)$ to $L'_3(k)$ for all k as follows. Let A be the oriented arc on T, which runs from one side of L_3 to the other and gives, along with the coherently oriented part of L_3, an oriented curve isotopic to the meridian of L_2.



Choose f to carry the curve L_3 on T to the (p, q) torus knot and A to the arc which joins two adjacent strings in the (p, q) knot as shown.



This homeomorphism f of the complement of L_1 and L_2 then carries each $L(k)$ to $L'(k)$.

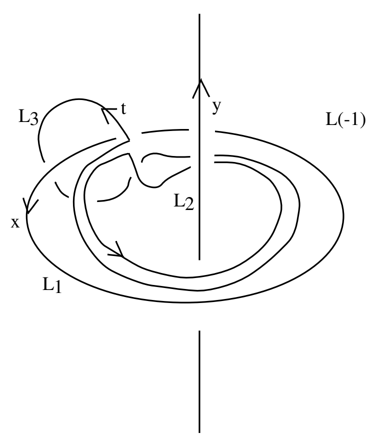
Now f is determined by its effect on the torus T, which is given by a 2×2 unimodular matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$. We can find r and s explicitly in terms of p and q, knowing that f carries the oriented graph $L_3 \cup A$ to the (p, q) torus knot L'_3 together with the arc between adjacent strings in its braid presentation. Following this oriented arc on T with the coherently oriented part of L'_3 gives a curve whose linking number with L_1 must lie between 1 and $p-1$, as it will form one component of a p-string closed braid with axis L_1 made from putting the half-twist in the adjacent strings. Since A together with the coherently oriented part of L_3 is isotopic to the meridian y of L_2, we know that f carries this to a curve whose linking number with L_1 is r. Consequently $0 < r < p$ (and $0 < s < q$). This determines r and s, since $s \equiv p^{-1} \bmod q$ and $r \equiv -q^{-1} \bmod p$.

To find the Alexander polynomial Δ'_k for the link $L'(k)$ with k half-twists it is enough to find the polynomial Δ_k for the link $L(k)$ and then substitute $f_*(x)$ and $f_*(y)$ for x and y.

In terms of the homology of $S^3 - L'$ the original meridian x becomes $f_*(x) = x^p y^q t^{pq}$ and y becomes $f_*(y) = x^r y^s t^{rq}$, since the image of the meridian x lies in the solid torus with core L_1 and represents q times the core, so its linking number with L'_3 is q times the linking number of L_1 with L'_3 giving the term t^{pq}, while the image of the meridian y represents r times the core of L_2, giving the term t^{rq}.

The basic link $L(0)$ has polynomial $\Delta_L(0) = 1 - x$, using for example the characteristic polynomial of the reduced Burau matrix for the identity braid on 2 strings (L_2 and L_3) with axis L_1. Substituting $f_*(x)$ for x gives $\Delta_{L'(0)} = 1 - x^p y^q t^{pq}$.

We already have $\Delta_0 = 1 - x$, so it is enough to find Δ_1 or Δ_2, or indeed Δ_{-1}. In fact $L(-1)$ is the fairly simple link shown here.



This yields $\Delta_{-1} = (1 - y)(1 - x(yt)^{-1})$, and gives

$$\Delta_1 - \Delta_0 = -t(\Delta_0 - \Delta_{-1}) = (1 + t)x - ty - xy^{-1}$$

and $\Delta_2 - \Delta_0 = (1 - t^2)x - t(1 - t)y - (1 - t)xy^{-1}$.

Then

$$\Delta_{2m} = 1 - t^{2m}x - (1-t)(1+t^2+\cdots+t^{2m-2})(ty+xy^{-1})$$

for $m > 0$, and so

$$\Delta'_{2m} = 1 - t^{2m}x^p y^q t^{pq} - (1-t)(1+t^2+\cdots+t^{2m-2})(x^r y^s t^{rq+1} + x^{p-r}y^{q-s}t^{(p-r)q}).$$

The corresponding formula for $m < 0$ is

$$\Delta_{-2m} = 1 - t^{-2m}x + (1-t)(t^{-2}+t^{-4}+\cdots+t^{-2m})(ty+xy^{-1})$$

giving

$$\Delta'_{-2m} = 1 - t^{-2m}x^p y^q t^{pq} + (1-t)(t^{-2}+t^{-4}+\cdots+t^{-2m})(x^r y^s t^{rq+1} + x^{p-r}y^{q-s}t^{(p-r)q}).$$

To find the Alexander polynomial of the (p,q) torus knot with $2m$ half-twists we apply f_* as above to get Δ'_{2m}, and then use the second general result which gives the Alexander polynomial of a sublink starting from the polynomial of the link.

Theorem 2 (Torres) *The Alexander polynomial of the sublink of L given by deleting a component L_1 with meridian x, leaving a link of more than one component, is found by setting $x = 1$ in Δ_L and dividing by $1 - X$, where the component L_1 represents X in the homology of the residual link $L - L_1$. If only one component remains, with meridian t, the Alexander polynomial of this knot is the expression above (which will be a rational function of t) multiplied by $1 - t$.*

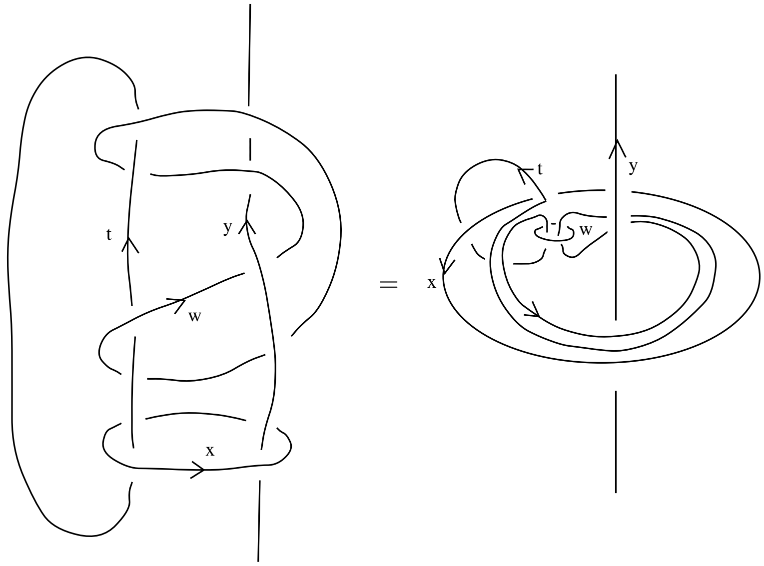
In our case, deleting both L_1 and L_2 from $L'(2m)$ will involve dividing Δ'_{2m} by $(1-t^p)(1-t^q)$ and multiplying by $1-t$, after setting $x = y = 1$.

Equivalently set $x = t^{pq}, y = t^{rq}$ in $\Delta_{2m}(1-t)/(1-t^p)(1-t^q)$ to get an explicit formula for the Alexander polynomial $\Delta(p,q,2m)$ for the (p,q) torus knot with $m > 0$ full twists in adjacent strings.

$$\begin{aligned}\Delta(p,q,2m) &= \frac{1-t}{(1-t^p)(1-t^q)} \\ &\times \ (1-(1-t)(1+t^2+\cdots+t^{2m-2})(t^{rq+1}+t^{(p-r)q}) - t^{pq+2m}).\end{aligned}$$

This form works well for $m \geq 0$, as it gives the Alexander polynomial as a genuine polynomial, with non-zero constant term. Indeed it is well-adapted for power series expansion. The two critical powers of t which contribute to the changes of the polynomial with m are $t^{rq+1} = t^{ps}$ and $t^{(p-r)q}$. If the roles of p and q are reversed then these terms change places, since $p - r \equiv q^{-1} \bmod p$ and $s \equiv p^{-1} \bmod q$. We shall assume that we have ordered p and q so that ps is the smaller of the two exponents. Equivalently we have arranged that $s < \frac{1}{2}q$ (and hence $r < \frac{1}{2}p$).

The formula for Δ_{2m} can be derived without using the recurrence relation from the multivariable polynomial of the 4-component link shown.



Using the presentation of this link as the closure of the braid

$$\sigma_3\sigma_2\sigma_1^2\sigma_2^{-2}\sigma_1^2\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^2\sigma_2,$$

its multivariable polynomial can be found using the multivariable Burau calculation procedure [5]. In terms of the meridians x, y, t, w, it is

$$(1-t^2)(1-xw)-(1-t)(1-w)(yt+xy^{-1}).$$

The polynomial for the link $L(2m)$ can then be derived, using theorems 1 and 2. First put m full twists on the two strings through the unknotted component with meridian w, where the effect on the polynomial, by theorem 1, is to replace w by wt^{2m}. Then delete this unknotted component leaving the link $L(2m)$. By theorem 2 the polynomial is then given by setting $w=1$ and dividing by $1-t^2$, to get

$$\Delta_{2m}=1-xt^{2m}-\frac{(1-t)(1-t^{2m})}{1-t^2}(yt+xy^{-1})$$

for all $m \in \mathbf{Z}$.

4 Sequences of polynomials whose coefficients are not all $0, \pm 1$

In this section we give conditions on $p, q > 0$ which ensure that the only possible Alexander polynomials in the sequence $\Delta(p, q, 2m)$ with all their coefficients $0, \pm 1$ are those with $|m| \leq 1$, and hence by [7] at most three knots in the sequence yield lens spaces after Dehn surgery.

We start with a result for the part of the sequence with $m \geq 0$.

Theorem 3 *Suppose that $s < \frac{1}{3}q$, where $s \equiv p^{-1} \bmod q$ and $0 < s < q$. Then the coefficient of t^{ps+2} in $\Delta(p, q, 2m)$ is ≤ -2 for all $m \geq 2$.*

For example, if $\{p,q\} = \{7,17\}$ we have $5 \equiv 7^{-1} \bmod 17$ and the coefficient of t^{37} is -2 for $m \geq 2$.

Proof. Under the given conditions $ps < (p-r)q$, and $p,q > 3$. For $m \geq 2$ the only terms that can contribute to t^{ps+2} are

$$\frac{1-t}{(1-t^p)(1-t^q)}(1-(1-t)(1+t^2)t^{ps}).$$

Expand $((1-t^p)(1-t^q))^{-1}$ as $(1+t^p+t^{2p}+\cdots)(1+t^q+t^{2q}+\cdots) = A(p,q)$, say. We must examine the coefficient of t^{ps+2} in $(1-t)A(p,q) - t^{ps}(1-t)^2(1+t^2)A(p,q)$. Now $(1-t)^2(1+t^2)A(p,q) = 1-2t+2t^2$ up to terms in t^2, and will contribute -2 to the coefficient of t^{ps+2}.

It is then enough to show that the coefficient of t^{ps+2} in $(1-t)A(p,q)$ is ≤ 0. This in turn will be guaranteed by showing that the coefficient of t^{ps+2} in $A(p,q)$ is zero. Now this coefficient counts the number of solutions of the equation $ap+bq = ps+2$ in non-negative integers a,b.

Since $ps \equiv 1 \bmod q$ we have $ap \equiv 3 \bmod q$ and so $3ps - ap \equiv 0 \bmod q$. Then $3s \equiv a \bmod q$, but this is not possible since $0 \leq a \leq s < 3s < q$, by hypothesis. □

The formula for the Alexander polynomial $\Delta(p,q,-2m)$ of the (p,q) torus knot with m negative full twists in adjacent strings (where $p,q>0$) is given from Δ'_{-2m} above as

$$\begin{aligned}\Delta(p,q,-2m) &= \frac{1-t}{(1-t^p)(1-t^q)} \\ &\times (1+(1-t)(t^{-2}+t^{-4}+\cdots+t^{-2m})(t^{rq+1}+t^{(p-r)q}) - t^{pq-2m}).\end{aligned}$$

This can be adapted for power series computation by considering

$$\begin{aligned}t^{2m}\Delta(p,q,-2m) &= \frac{1-t}{(1-t^p)(1-t^q)} \\ &\times (t^{2m}+(1-t)(1+t^2+\cdots+t^{2m-2})(t^{rq+1}+t^{(p-r)q}) - t^{pq}).\end{aligned}$$

Again we shall assume that we have ordered p and q so that ps is the smaller of the two critical powers $rq+1 = ps$ and $(p-r)q$ of t which contribute to the changes with m.

The following general result for negative twists complements the previous result, under the same conditions.

Theorem 4 *Suppose that $s < \frac{1}{3}q$, where $s \equiv p^{-1} \bmod q$ and $0 < s < q$. Then the coefficient of at least one of the terms $t^{ps+1}, t^{ps+2}, t^{ps+3}$ in $t^{2m}\Delta(p,q,-2m)$ is ± 2 for all $m \geq 2$.*

Proof. Under the given conditions $ps < (p-r)q$, and $p,q > 3$. For $m \geq 2$ we have

$$t^{2m}\Delta(p,q,-2m) = \frac{1-t}{(1-t^p)(1-t^q)}(t^{2m}+(1-t)(1+t^2)t^{ps})$$

up to terms in t^{ps+3}. Expand $((1-t^p)(1-t^q))^{-1}$ as $A(p,q)=\sum a_i t^i$, where a_i counts the number of ways to write $i=ap+bq$ with non-negative integers a,b. For $i\le pq$ we know that $a_i=0$ or 1. Furthermore, $i=ps$ is the first time that two consecutive coefficients a_{i-1} and a_i are both 1, as $s\equiv p^{-1} \bmod q$.

Since we have assumed that $3s<q$ it also follows that we can't have $a_i=a_{i+2}=1$ with $i<ps$. Thus in any four consecutive coefficients of $\sum_{i=0}^{ps} a_i t^i$ there are two consecutive coefficients which are equal (either to 0, or 1), and so among any 3 consecutive coefficients of $(1-t)\sum_{i=0}^{ps} a_i t^i$ at least one of them is zero.

Now consider the coefficients of the three consecutive terms $t^{ps+1}, t^{ps+2}, t^{ps+3}$ in $t^{2m}\Delta(p,q,-2m)$. The contribution from $(1-t)^2(1+t^2)A(p,q)t^{ps}$ is $(1-2t+2t^2-2t^3)t^{ps}$, while the contribution from $t^{2m}(1-t)A(p,q)$ involves three consecutive coefficients of $(1-t)A(p,q)$ up to degree at most ps. At least one of these must be zero, leaving one of the coefficients as ± 2.

(Of course, once $2m>ps+1$ the coefficient of t^{ps+1} will be -2, and the lowest degree term in the whole polynomial will be t^{ps} so that in standard polynomial form the Alexander polynomial is $1-2t+\cdots$.) $\square$

5 Some contrasting examples.

The conditions on p and q in theorems 3 and 4 can be phrased simply in terms of the continued fraction expansion of $p/q=[a_0,a_1,\ldots,a_k]=a_0+1/(a_1+1/(\cdots+1/a_k))$, where each $a_i\ge 1$ and $a_k\ge 2$.

Definition. A Laurent polynomial with integer coefficients is *thick* if it has some coefficient a with $|a|>1$.

A knot whose Alexander polynomial is thick admits no lens space surgery, [7].

Theorem 5 (rephrasing theorems 3, 4) *If $p,q>3$ and $p/q=[a_0,a_1,\ldots,a_k]$ with $a_k\ge 3$ then $\Delta(p,q,2m)$ is thick for all m with $|m|\ge 2$.*

Proof. If $p>q$ and $p/q=[a_0,a_1,\ldots,a_k]$ then $q/p=[0,a_0,a_1,\ldots,a_k]$. We can then assume, by swapping p and q if necessary, that k is odd. Then

$$\begin{pmatrix}1 & a_0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ a_1 & 1\end{pmatrix}\cdots\begin{pmatrix}1 & 0\\ a_k & 1\end{pmatrix}=\begin{pmatrix}p & r\\ q & s\end{pmatrix}$$

where $0<r<p$ and $0<s<q$. Hence $\begin{pmatrix}p & r\\ q & s\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ a_k & 1\end{pmatrix}$ for some non-negative a,b,c,d. It follows that $q=c+da_k>da_k=sa_k$ unless $s=1$ and $q=a_k$. When $a_k\ge 3$ and $p,q>3$ we have $3s<q$ as required for theorems 3 and 4. $\square$

The methods in theorem 4 show also that, except in the case $q=2$, when the term $t^{(p-r)q}$ also contributes to the coefficient of t^{ps+1}, the Alexander polynomial $\Delta(p,q,-2m)$ will start $1-2t+\cdots$ for sufficiently large m.

In contrast to this if $(p-1)(q-1)<2ps<pq$ then all the knots with $2m>0$ half-twists have coefficients $0,\pm 1$. This follows since the adjustments in the series in passing from $2m$ to $2m+2$ occur after the half-way stage $(p-1)(q-1)/2$ in the Alexander polynomial, and inductively all the terms must be $0, \pm 1$, by symmetry of the Alexander polynomial.

This happens, when $q=3$, and in some cases when $a_k = 2$, for example when $p \equiv \pm 2 \bmod q$. This includes the case $(5,8)$ but not the next Fibonacci pair $(8,13)$, which has some some coefficients ± 2 for certain positive values of m.

Noting that the cases where $m = \pm 1$, $q = 3$ or $q = \pm 2 \bmod p$ are those which satisfy Dean's primitive/Seifert fibred condition it is interesting to speculate on how far this condition identifies knots with thin Alexander polynomial among Dean's general twisted torus knots.

Acknowledgements

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